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1) $y = e^{ax} \cdot \sin x$, find $y_n = ?$

$$\rightarrow y_1 = e^{ax} \cdot \cos x \cdot a \cdot 1$$

$$y_2 = a^2 \cdot e^{ax} (-\sin x) \cdot 1$$

$$y_3 = e^{ax} \cdot a^3 \cdot (-\cos x) \cdot 1$$

$$y_4 = e^{ax} \cdot a^4 (\sin x) \cdot 1$$

similarly moving upto 'n' terms, we have

$$y_n = e^{ax} \cdot a^n \cdot 1^n \cdot \sin\left(\frac{n\pi}{2} + x\right)$$

$$\therefore y_n = a^n e^{ax} \sin\left(\frac{n\pi}{2} + x\right)$$

2) $u = \tan^{-1} \left(\frac{x^3 + y^3}{(x-y)} \right)$ show $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \sin 2u$

Since, function is non-homogeneous,

$$\Rightarrow \tan u = \frac{x^3 + y^3}{(x-y)}$$

, upon taking x^3 from Nr. and x from Dr. in R.H.S outside we have

$$\Rightarrow \tan u = \frac{x^3}{x} \left\{ \frac{1 + (y/x)^3}{1 - (y/x)} \right\}$$

$$\Rightarrow \tan u = x^2 \left\{ \frac{1 + (y/x)^3}{1 - (y/x)} \right\}$$

now the function $\tan u$ is homogeneous, so we can now apply Euler's theorem in it.

as per Euler's theorem,

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = nv$$

where v is a homogeneous function of degree ' n '.

we have the function ' $\tan u$ ', a homogeneous function with degree 2,

$\Rightarrow$ upon partial differentiation w.r. to x and y respectively we have

$$\frac{\partial v}{\partial x} = \frac{\partial \tan u}{\partial x} = \sec^2 u \cdot \frac{\partial u}{\partial x} \rightarrow \textcircled{i}$$

$$\frac{\partial v}{\partial y} = \frac{\partial \tan u}{\partial y} = \sec^2 u \cdot \frac{\partial u}{\partial y} \rightarrow \textcircled{ii}$$

multiplying $\textcircled{i}$ and $\textcircled{ii}$ with x and y respectively, we have

$$\begin{aligned} x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} &= x \sec^2 u \cdot \frac{\partial u}{\partial x} + \sec^2 u \frac{\partial u}{\partial y} \cdot y \\ &= nv \quad [\text{Euler's theorem}] \end{aligned}$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{2x \tan u}{\sec^2 u} \quad \left[\begin{array}{l} \text{taking } \sec^2 u \\ \text{common} \end{array} \right]$$

$$\Rightarrow x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 2x \sin u \cos u = \sin 2u$$

12 ✓ proved.

③ Maclaurin's theorem, states that for a function $f(x)$ in $[0, x]$ when completely differentiable in all its range can be expanded as per given series.

$$f(x) = f(0) + \underbrace{f'(0)}_{[1]}x + \underbrace{f''(0)}_{[2]}x^2 + \underbrace{f'''(0)}_{[3]}x^3 + \dots + \underbrace{f^{(n)}(0)}_{[n]}x^n$$

[Note $\rightarrow$ here superscript denotes the derivatives]

proof: assuming a function $f(x)$ in range $[0, x]$, it can be expanded in series as shown below

$$f(x) = A_0 + A_1x + A_2x^2 + \dots + A_nx^n$$

where $A_0, A_1, A_2, \dots, A_n$ are all independent of x .

$\rightarrow$ putting $x=0$ in $f(x)$ we have $A_0 = 0$

$\rightarrow$ upon differentiating $f(x)$, we have

$$f'(x) = 0 + A_1 + 2xA_2 + \dots + A_n \cdot nx^{n-1}$$

putting $x=0$, we have in $f'(x)$,

$$f'(0) = A_1$$

similarly following the same procedure of differentiating the function $f(x)$ and initialising $x=0$ each time we have

$$f''(0) = 2A_2 = 12A_2$$

$$f'''(0) = 6A_3 = 12A_3$$

$$f^{n\text{th}}(0) = n! A_n$$

obtaining the values of A_1, A_2, A_3, A_4 we have

$$A_1 = \frac{f'(0)}{1!} \quad ; \quad A_2 = \frac{f''(0)}{2!}$$

$$A_3 = \frac{f'''(0)}{3!} \quad \text{and so on} \dots$$

$$\text{where } A_n = \frac{f^{n\text{th}}(0)}{n!}$$

putting the values of $A_1, A_2, \dots, A_n$ in the eqn $f(x)$
 "Maclaurin's theorem" will be proved as
 done below

$$f(x) = f(0) + A_1 x + A_2 x^2 + A_3 x^3 + \dots + A_n x^n$$

$$\therefore f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots + \frac{f^{n\text{th}}(0)}{n!} x^n$$

Hence Maclaurin's theorem is proved.

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(4) $y^{1/m} + y^{-1/m} = 2x$ procure $(x^2-1)y_2 + xy_1 - m^2y = 0$

$\Rightarrow$ taking $y^{1/m} = t$

$\Rightarrow y = t^m$

$\Rightarrow y^{-1/m} = t^{-m} \rightarrow (i)$

so, the eqn. becomes, $\Rightarrow \left[t + \frac{1}{t} = 2x \right]$

$t + \frac{1}{t} = 2x$

$\Rightarrow t^2 + 1 - 2xt = 0$

$\Rightarrow$ solving the quadratic w.r. to x ,

$\Rightarrow t = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$

$\Rightarrow t = x \pm \sqrt{(x^2-1)}$

$\Rightarrow t \cdot y^{1/m} = x \pm \sqrt{(x^2-1)}$

$\Rightarrow$ eqn. becomes,

$\rightarrow x + \sqrt{(x^2-1)} + \frac{1}{x + \sqrt{(x^2-1)}} = 2x$

$\Rightarrow \left(x + \sqrt{(x^2-1)} \right)^2 + 1 - 2x \left(x + \sqrt{(x^2-1)} \right) = 0$

$\Rightarrow x^2 + (x^2-1) + 2x\sqrt{(x^2-1)} + 1 - 2x^2 - 2x\sqrt{(x^2-1)} = 0 \Rightarrow 0 = 0$

$\Rightarrow$

$$y = t^m \quad (\text{from eqn (i)})$$

$$y = \left[x + \sqrt{x^2 - 1} \right]^m, \text{ upon differentiation}$$

$$y_1 = m \left[x + \sqrt{x^2 - 1} \right]^{m-1} \cdot \left[1 + \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x \right]$$

$$= m \left[x + \sqrt{x^2 - 1} \right]^{m-1} \cdot \left[\frac{\sqrt{x^2 - 1} + x}{(\sqrt{x^2 - 1})} \right]$$

$$= m \left[x + \sqrt{x^2 - 1} \right]^m \times \frac{1}{\sqrt{x^2 - 1}}$$

$$y_1 = m y \times \frac{1}{\sqrt{x^2 - 1}} \rightarrow \textcircled{\text{ii}}$$

upon squaring the eqn (ii) we have

$$\Rightarrow y_1^2 (x^2 - 1) = m^2 y^2 \rightarrow \textcircled{\text{iii}}$$

upon further differentiation of eqn (iii) we have

$$\Rightarrow 2y_1 (x^2 - 1) y_2 + 2x (y_1^2) + 0 = m^2 y_1 \times 2y$$

$$\Rightarrow 2y_1 (x^2 - 1) y_2 + 2x y_1^2 - m^2 y_1 \cdot 2y = 0$$

upon eliminating $2y_1$ we have

$$\Rightarrow (x^2 - 1) y_2 + x y_1 - m^2 y = 0 \quad \text{proved}$$

⑤ ⑦ given: $3x + 4y = 5$

let $u = 3x + 4y - 5$

now, differentiating u partially w.r. to x and y respectively we have

$$\frac{\partial u}{\partial x} = 3 + 4y$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3 + 4y + 4 + 3x$$

$$\frac{\partial u}{\partial y} = 4 + 3x$$

$$= 7 + 4y + 3x$$

$$\frac{\partial^2 u}{\partial x^2} =$$

$\Rightarrow$ differentiating $3x + 4y = 5$ w.r. to x and y partially and respectively we have

$$3 + 4y = 0$$

$$y = -3/4$$

$$3x + 4 = 0$$

$$x = -4/3$$

to obtain the

$$\textcircled{c} \quad f(x, y) = 2x^2 + 2y^2 - xy - 20x.$$

→ upon partial differentiation of the given fn w.r. to x and y respectively we have

$$\frac{\partial u}{\partial x} = 4x + 0 - y - 20 \rightarrow \textcircled{i}$$

$$\frac{\partial u}{\partial y} = 4y + 0 - x - 20 \rightarrow \textcircled{ii}$$

initialising eqn $\textcircled{i}$ & $\textcircled{ii}$ to zero; we have

$$4x - y - 20 = 0$$

$$\Rightarrow y = 4x - 20 \rightarrow \textcircled{iii}$$

$$4y - x - 20 = 0.$$

$$x = 4y - 20 \rightarrow \textcircled{iv}$$

now we conclude that

$-x = y$ satisfy both the eqns. $\textcircled{iii}$ and $\textcircled{iv}$ at a particular point

$$\frac{\partial^2 u}{\partial x^2} = 4$$

$$\frac{\partial (\partial u)}{\partial x \partial y} = -1$$

$$\frac{\partial^2 u}{\partial y^2} = 4$$

$$\text{now, } D = \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left[\frac{\partial (\partial u)}{\partial x \partial y} \right]^2$$

$$D = 16 - 1^2 > 0;$$

So, value of max. or min exist,

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$$\frac{\partial^2 u}{\partial x^2} = 4 > 0$$

so only value of ~~max~~
minima exist, at
point $(4, -4)$.

from eqn. (i) & (ii) we have

$$u = 4(4u - 20) - 20$$

$$\Rightarrow u = 16u - 60$$

$$\Rightarrow u = 4$$

$$\text{and } y = -4$$

$\Rightarrow$ putting $u = 4$ in eqn (iii) we have

$$y = 16 - 20 = -4$$

$$(4) (5) \quad y = \sin 3x - 3 \sin x$$

$$\begin{aligned} y_1 &= \cos 3x \cdot 3 - 3 \cos x \\ &= 3 [\cos 3x - \cos x] \\ &= 3 [-\sin 2x \sin x] \end{aligned}$$

$$\cos C - \cos D = -\sin\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

$$y_1 = -3 \sin 2x \sin x$$

putting $y_1 = 0$ we have

$$\sin 2x \cdot \sin x = 0$$

$$\sin 0 = 0$$

$$x = 0$$

$$y_2 = -3 [2 \sin 2x \cdot \sin x + \cos x \sin 2x]$$

$$y_2 = 0$$

$\Rightarrow$ at point $(0, 0)$ we obtain

$$\textcircled{1} \quad y = e^{ax} \sin bx, \quad y'' = ?$$

$$\rightarrow y_1 = a e^{ax} b \cos bx$$

$$\rightarrow y_2 = a^2 e^{ax}$$

$$\textcircled{1} \quad y = e^{ax} \sin bx ;$$

$\Rightarrow$ applying U.V rule,

$$\Rightarrow y_1 = a e^{ax} \sin bx + e^{ax} b \cos bx$$

$$y_1 = e^{ax} [a \sin bx + b \cos bx]$$

$$\left[\begin{aligned} \sin(A+B) &= \sin A \cos B \\ &+ \cos A \sin B \end{aligned} \right]$$

$$r = \sqrt{a^2 + b^2}$$

let $b = r \sin \theta$ and $a = r \cos \theta$, we have

$$\Rightarrow y_1 = e^{ax} [r \sin bx \cos \theta + r \cos bx \sin \theta]$$

$$= r e^{ax} [\sin(bx + \theta)]$$

$$\Rightarrow y_2 = r [e^{ax} b \cos(bx + \theta) + a e^{ax} \sin(bx + \theta)]$$

similarly $b = r \sin \theta$ and $a = r \cos \theta$, y_2 will be

$$\rightarrow y_2 = r^2 [e^{ax} \cos(bx + \theta) \sin \theta + \sin(bx + \theta) \cos \theta]$$

$$= r^2 [e^{ax} \sin(bx + \theta + \theta)]$$

$$= r^2 [e^{ax} \sin(bx + 2\theta)]$$

Similarly moving upto n th term, we have

$$y_n = r^n [e^{ax} \sin(bx + n\theta)]$$

now

$$r^2 = (a^2 + b^2)$$

$$\Rightarrow r^n = (a^2 + b^2)^{n/2}$$

and $\theta = \tan^{-1} \frac{b}{a}$

$$\frac{a}{b} = \frac{r \cos \theta}{r \sin \theta}$$

$$\Rightarrow \theta = \tan^{-1} \frac{b}{a}$$

$$\Rightarrow y_n = (a^2 + b^2)^{n/2} [e^{ax} \sin(bx + n \tan^{-1} \frac{b}{a})]$$

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